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Linearized solution of a flow over a nonuniform bottom

Y.Z. BOUTROS, M.B. ABD-EL-MALEK and S.Z. MASOUD
Department of Engineering Mathematics and Physics, Faculty of Engineering, Alexandria University, Alexandria, Egypt

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Abstract: A linearized theory is presented for determining the shape of the free surface of a running stream which is disturbed by some irregularities lying on the bottom. The bottom is represented in integral form using Fourier's double-integral theorem. Then following Lamb [3], a linear free-surface profile is obtained for the supercritical and subcritical cases.

The results are plotted for the two cases of the flow for different shapes of the bottom, and different values of the Froude number. The effect of the Froude number, the bottom height and the shape of the bottom are discussed.

1. Introduction

Fluid flow over various bottom topographies has attracted considerable attention throughout the history of fluid mechanics, and the literature on the topic is extensive. In 1900, Wien [2] suggested a linearized solution for a flow over a small step. In 1932, Lamb [3] presented a general linearized theory for flow over stream beds of arbitrary shape. Lamb's theory was reviewed by Wehausen and Laitone [4], who also discussed the free-surface flow over a step discontinuity in the stream bed. The book by Kochin, Kibel and Rozen [5] contains detailed and elegant solutions for the cases of a point vortex, a point source and a dipole moving beneath the surface of an infinitely deep fluid. Recently, a linearized theory for flow over certain bottom profiles was developed by Gazdar [6]. Forbes [7] investigated the flow over a submerged semi-elliptical body. Abd-el-Malek [8] solved the problem of a flow over a ramp for the nonlinear case applying Hilbert's transformation. Forbes and Schwartz [9] considered the case of flow over a semi-circular obstruction. Forbes [10] investigated the flow over a semi-circular obstruction including the influence of gravity and surface tension. In this paper, we present the linear solution of a two-dimensional, steady, inviscid, incompressible, and irrotational flow over an infinite open channel with a nonuniform bottom.

2. Formulation of the problem

Consider the steady, two-dimensional flow of an ideal fluid in an infinite open channel with a nonuniform bottom as shown in Fig. 1.

The flow far upstream is uniform with velocity U in the positive x direction, and depth h. Lamb [3] assumed the case of a simple harmonic corrugation given by

$$y = -h + a \cos kx, \tag{2.1}$$

the origin being in the undisturbed surface. He found that the free surface profile is

$$\eta = \frac{a \cos kx}{\cosh kh - (g/kU^2) \sinh kh}. \tag{2.2}$$

This solution may be generalized by Fourier's theorem. In our problem, the bottom is represented in integral form using Fourier's theorem.

The profile of the bed will take the form

$$y = -h + \frac{1}{\pi}\int_0^\infty \mathrm{d}k \int_{-\infty}^\infty f(s) \cos k(x-s)\, \mathrm{d}s. \tag{2.3}$$

Then, the free surface will be obtained by superposition of terms of the type (2.2) due to various elements of the Fourier-integral; thus,

$$\eta = \frac{1}{\pi}\int_0^\infty \mathrm{d}k \int_{-\infty}^\infty \frac{f(s) \cos k(x-s)}{\cosh kh - (g/kU^2) \sinh kh}\, \mathrm{d}s. \tag{2.4}$$

3. Analysis

From Fig. 1 we find that

$$Y(x) = \begin{cases} 0, & x < 0, \\ x \tan \alpha, & 0 < x < l \cot \alpha, \\ -x \tan \beta + l(\cot \alpha \tan \beta + 1), & l \cot \alpha < x < l(\cot \alpha + \cot \beta), \\ 0, & x > l(\cot \alpha + \cot \beta). \end{cases} \tag{3.1}$$

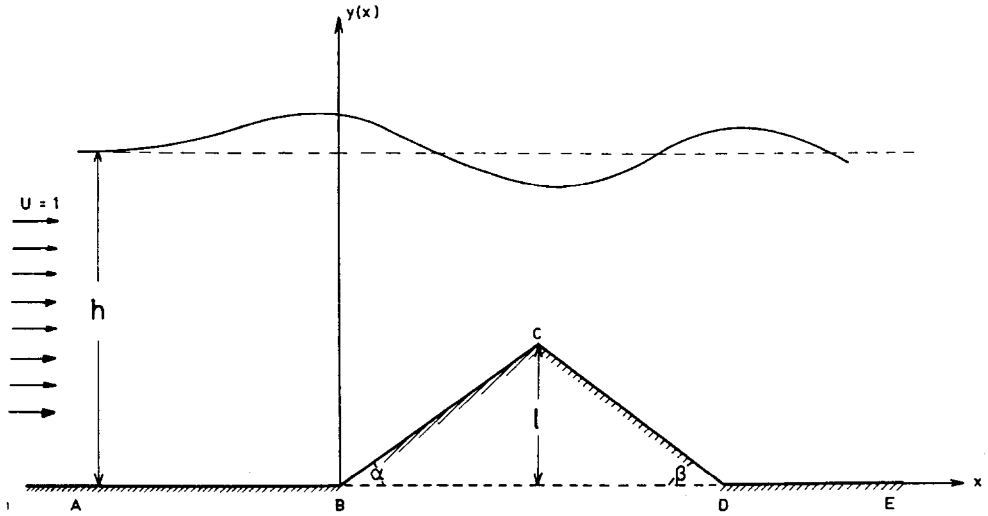

Fig. 1. The physical problem for a flow over a nonuniform bottom.

To express $f(x)$ in integral form we apply Fourier's double integral theorem

$$Y(x)=\frac{1}{\pi}\int_0^\infty \mathrm{d}k\int_{-\infty}^{\infty} Y(s)\cos k(x-s)\,\mathrm{d}s. \tag{3.2}$$

Substituting (3.1) into (3.2) we get

$$\begin{aligned}Y(x)&=\frac{1}{\pi}\int_0^\infty \mathrm{d}k\int_0^{l\cot\alpha} s\tan\alpha\cos k(x-s)\,\mathrm{d}s\\&\quad+\frac{1}{\pi}\int_0^\infty \mathrm{d}k\int_{l\cot\alpha}^{l(\cot\alpha+\cot\beta)}(l\cot\alpha\tan\beta+l-s\tan\beta)\cos k(x-s)\,\mathrm{d}s.\end{aligned} \tag{3.3}$$

Hence the shape of the bed will take the form

$$\begin{aligned}Y(x)&=\frac{\tan\alpha}{\pi}\int_0^\infty\frac{1}{k^2}[\cos k(x-l\cot\alpha)-\cos kx]\,\mathrm{d}k\\&\quad-\frac{\tan\beta}{\pi}\int_0^\infty\frac{1}{k^2}[\cos k(x-l\cot\alpha-l\cot\beta)-\cos k(x-l\cot\alpha)]\,\mathrm{d}k.\end{aligned} \tag{3.4}$$

If the depth of the channel is finite and equal to h, taking the origin in the undisturbed surface, we may write the shape of the bed in the form

$$\begin{aligned}Y(x)&=-h+\int_0^\infty a_1(k)\cos x_2k\,\mathrm{d}k-\int_0^\infty a_1(k)\cos x_1k\,\mathrm{d}k\\&\quad-\int_0^\infty a_2(k)\cos x_3k\,\mathrm{d}k+\int_0^\infty a_2(k)\cos x_2k\,\mathrm{d}k,\end{aligned} \tag{3.5}$$

where

$$a_1(k)=\tan\alpha/k^2,\qquad a_2(k)=\tan\beta/k^2, \tag{3.6}$$

and

$$x_1=x,\quad x_2=x-1\cot\alpha,\quad x_3=x-1(\cot\alpha+\cot\beta). \tag{3.7}$$

Then, following Thomson [1], and Lamb [3], the shape of the free surface is

$$\begin{aligned}y_1(x)\\&=\int_0^\infty\frac{a_1(k)\cos x_2k}{\cosh kh[1-\tanh kh/F^2kh]}\,\mathrm{d}k-\int_0^\infty\frac{a_1(k)\cos x_1k}{\cosh kh[1-\tanh kh/F^2kh]}\,\mathrm{d}k\\&\quad-\int_0^\infty\frac{a_2(k)\cos x_3k}{\cosh kh[1-\tanh kh/F^2kh]}\,\mathrm{d}k+\int_0^\infty\frac{a_2(k)\cos x_2k}{\cosh kh[1-\tanh kh/F^2kh]}\,\mathrm{d}k,\end{aligned} \tag{3.8}$$

where $F^2=(U^2/hg)$ is the Froude number. Introducing the quantities

$$t=kh,\qquad X_1=x_1/h,\quad X_2=x_2/h,\quad X_3=x_3/h, \tag{3.9}$$

and

$$\eta=y_1/h,$$

we have the normalized shape of the free surface

$$\eta(x)=\frac{\tan\alpha}{\pi}\int_0^\infty \frac{\cos X_2t-\cos X_1t}{t^2[\cosh t-\sinh t/F^2t]}\,\mathrm{d}t$$
$$-\frac{\tan\beta}{\pi}\int_0^\infty \frac{\cos X_3t-\cos X_2t}{t^2[\cosh t-\sinh t/F^2t]}\,\mathrm{d}t. \tag{3.10}$$

Now consider the complex function

$$G(z)=z^2[\cosh z-\sinh z/F^2z], \tag{3.11}$$

which is the common denominator of the integrands in (3.10).

To evaluate the integrals in (3.10) we use a well-known analytical method of Cauchy which is to expand $1/G(z)$, given in (3.11), in an infinite series of partial fractions in terms of the zeros of $G(z)$.

4. Case of the supercritical flow: $F^2>1$

The zeros of $G(z)$ are

$$\left.\begin{aligned} &z_0=0,\\ &z_{\pm n}=\pm\mathrm{i}y_n,\quad n=1,2,3,\ldots,\end{aligned}\right\} \tag{4.1}$$

where y_n are the positive roots of

$$\tan y=F^2y. \tag{4.2}$$

$G(z)$ is an even function and, therefore, we can write

$$\frac{1}{z^2(\cosh z-\sinh z/F^2z)}=\frac{A_0}{z^2}+\sum_{n=1}^{\infty}\frac{A_n}{z^2+y_n^2}.$$

From which we get

$$\frac{1}{z^2[\cosh z-\sinh z/F^2z]}=\frac{F^2}{F^2-1}\frac{1}{z^2}+2F^2\sum_{n=1}^{\infty}\frac{\cos y_n}{(F^2\cos^2y_n-1)(z^2+y_n^2)}. \tag{4.3}$$

Therefore, the surface shape is

$$\eta(x)=\frac{\tan\alpha}{\pi}\int_0^\infty\left[\frac{F^2}{(F^2-1)}\frac{1}{t^2}+2F^2\sum_{n=1}^{\infty}\frac{\cos y_n}{(F^2\cos^2y_n-1)(t^2+y_n^2)}\right]$$
$$\times[\cos X_2t-\cos X_1t]\,\mathrm{d}t$$
$$-\frac{\tan\beta}{\pi}\int_0^\infty\left[\frac{F^2}{(F^2-1)}\frac{1}{t^2}+2F^2\sum_{n=1}^{\infty}\frac{\cos y_n}{(F^2\cos^2y_n-1)(t^2+y_n^2)}\right]$$
$$\times[\cos X_3t-\cos X_2t]\,\mathrm{d}t. \tag{4.4}$$

Using the integral

$$\int_0^\infty\frac{p\cos kt}{t^2+p^2}\,\mathrm{d}t=\tfrac{1}{2}\pi\,\mathrm{e}^{-pk},$$

we have

$$\int_0^\infty \frac{\cos xt}{t^2+y_n^2}\,\mathrm{d}t = \tfrac{1}{2}\pi/y_n\, \mathrm{e}^{-|x|y_n}.$$

Moreover, we have

$$\int_0^\infty \frac{\cos X_1 t - \cos X_2 t}{t^2+y_n^2}\,\mathrm{d}t = \tfrac{1}{2}\pi/y_n\left(\mathrm{e}^{-|X_1|y_n} - \mathrm{e}^{-|X_2|y_n}\right).$$

Hence,

$$\int_0^\infty \frac{\cos X_1 t - \cos X_2 t}{t^2}\,\mathrm{d}t = \tfrac{1}{2}\pi\left(|X_2| - |X_1|\right).$$

Substituting in (4.4), we get

$$\begin{aligned}\eta(x) = {} & \frac{F^2 \tan\alpha}{2(F^2-1)}\left(|X_1| - |X_2|\right) \\ & + F^2 \tan\alpha\left\{\sum_1^\infty \frac{\cos y_n}{y_n\left(F^2\cos^2 y_n - 1\right)}\left(\mathrm{e}^{-|X_2|y_n} - \mathrm{e}^{-|X_1|y_n}\right)\right\} \\ & + \frac{F^2 \tan\beta}{2(F^2-1)}\left(|X_3| - |X_2|\right) \\ & + F^2 \tan\beta\left\{\sum_1^\infty \frac{\cos y_n}{y_n\left(F^2\cos^2 y_n - 1\right)}\left(\mathrm{e}^{-|X_2|y_n} - \mathrm{e}^{-|X_3|y_n}\right)\right\}.\end{aligned} \tag{4.5}$$

5. Case of the critical flow: $F^2 = 1$

For the critical case $F^2 = 1$, there is no solution, since $\int_0^\infty (\cos xt)/t^4\,\mathrm{d}t$ becomes unbounded due to a singularity of fourth order, in the function $G(z)$ given by (3.11), at $z = 0$.

6. Case of the subcritical flow: $F^2 < 1$

In this case $G(z)$ has a pair of real zeros $(\pm r)$ in addition to those of (4.1) and hence its zeros are

$$z_0 = 0, \quad z_{\pm 1} = \pm r, \qquad z_{\pm n} = \pm \mathrm{i} y_n, \quad n = 2, 3, 4, \ldots \tag{6.1}$$

where r is the positive root of the equation

$$\tanh x = F^2 x, \tag{6.2}$$

and y_n are the positive roots of

$$\tan x = F^2 x. \tag{6.3}$$

Then, we can write

$$\frac{1}{z^2[\cosh z - \sinh z/F^2 z]} = \frac{A_0}{z^2} + \frac{A_1}{z^2 - r^2} + \sum_{n=2}^{\infty} \frac{A_n}{z^2 + y_n^2},$$

from which we get

$$\frac{1}{z^2[\cosh z - \sinh z/F^2 z]} = \frac{F^2}{F^2 - 1}\frac{1}{z^2} + \frac{2F^2 \cosh r}{F^2 \cosh^2 r - 1}\frac{1}{z^2 - r^2} + 2F^2 \sum_{n=2}^{\infty} \frac{\cos y_n}{(F^2 \cos^2 y_n - 1)(z^2 + y_n^2)}. \tag{6.4}$$

Substituting in (3.10), we get the surface shape

$$\begin{aligned}\eta(x) = {} & \tan\alpha \int_0^{\infty} \left[\frac{F^2}{2(F^2-1)t^2} + \frac{F^2 \cosh r}{F^2 \cosh^2 r - 1}\frac{1}{t^2 - r^2} + F^2 \sum_{n=2}^{\infty} \frac{\cos y_n}{(F^2 \cos^2 y_n - 1)(t^2 + y_n^2)}\right](\cos X_2 t - \cos X_1 t)\,\mathrm{d}t \\ & + \tan\beta \int_0^{\infty} \left[\frac{F^2}{2(F^2-1)t^2} + \frac{F^2 \cosh r}{F^2 \cosh^2 r - 1}\frac{1}{t^2 - r^2} + F^2 \sum_{n=2}^{\infty} \frac{\cos y_n}{(F^2 \cos^2 y_n - 1)(t^2 + y_n^2)}\right](\cos X_2 t - \cos X_3 t)\,\mathrm{d}t.\end{aligned} \tag{6.5}$$

Now consider the function

$$f(t) = \cos xt/(t^2 - u^2).$$

This function possesses two singularities at $t = u$ and $t = -u$. Thus it is necessary to interpret the integral $\int_{-\infty}^{\infty} f(t)\,\mathrm{d}t$ as a contour integral in the complex t-plane, by taking the path of integration by passing the poles at u and $-u$ as two semicircular paths of vanishingly small radii. In this case we have

$$\int_0^{\infty} \frac{\cos xt}{t^2 - u^2}\mathrm{d}t = \frac{1}{2}\left\{ ⨍_0^{\infty} \frac{\cos xt}{t^2 - u^2}\mathrm{d}t - \frac{\pi}{u}\sin xu \right\}.$$

Far upstream, the two terms within the brackets cancel, so that the deformation of the surface is insensible. Far downstream, however, the terms within the brackets, summing to twice the value of the second term, i.e.

$$\int_0^{\infty} \frac{\cos xt}{t^2 - u^2}\mathrm{d}t = \begin{cases} -\dfrac{\pi}{u}\sin xu, & x > 0, \\ 0, & x < 0. \end{cases}$$

Hence the surface shape will take the form

$$\eta(x) = \frac{F^2 \tan\alpha}{2(F^2-1)}(|X_1|-|X_2|)$$

$$+F^2 \tan\alpha\left\{\sum_{n=2}^{\infty} \frac{\cos y_n}{y_n(F^2\cos^2 y_n - 1)}(\mathrm{e}^{-|X_2|y_n} - \mathrm{e}^{-|X_1|y_n})\right\}$$

$$+\frac{F^2 \tan\beta}{2(F^2-1)}(|X_3|-|X_2|)$$

$$+F^2 \tan\beta\left\{\sum_{n=2}^{\infty} \frac{\cos y_n}{y_n(F^2\cos^2 y_n - 1)}(\mathrm{e}^{-|X_2|y_n} - \mathrm{e}^{-|X_3|y_n})\right\}, \quad x<0; \tag{6.6a}$$

and

$$\eta(x) = \frac{F^2 \tan\alpha}{2(F^2-1)}(|X_1|-|X_2|) + F^2 \tan\alpha \frac{\cosh r}{r(F^2\cosh^2 r - 1)} \sin rX_1$$

$$+F^2 \tan\alpha\left\{\sum_{n=2}^{\infty} \frac{\cos y_n}{y_n(F^2\cos^2 y_n - 1)}(\mathrm{e}^{-|X_2|y_n} - \mathrm{e}^{-|X_1|y_n})\right\}$$

$$+\frac{F^2 \tan\beta}{2(F^2-1)}(|X_3|-|X_2|)$$

$$+F^2 \tan\beta\left\{\sum_{n=2}^{\infty} \frac{\cos y_n}{y_n(F^2\cos^2 y_n - 1)}(\mathrm{e}^{-|X_2|y_n} - \mathrm{e}^{-|X_3|y_n})\right\},$$

$$0 < x < l\cot\alpha; \tag{6.6b}$$

$$\eta(x) = \frac{F^2 \tan\alpha}{2(F^2-1)}(|X_1|-|X_2|) + F^2 \tan\alpha\left[\frac{\cosh r}{r(F^2\cosh^2 r - 1)}(\sin rX_1 - \sin rX_2)\right]$$

$$+F^2 \tan\alpha\left\{\sum_{n=2}^{\infty} \frac{\cos y_n}{y_n(F^2\cos^2 y_n - 1)}(\mathrm{e}^{-|X_2|y_n} - \mathrm{e}^{-|X_1|y_n})\right\}$$

$$+\frac{F^2 \tan\beta}{2(F^2-1)}(|X_3|-|X_2|)$$

$$+F^2 \tan\beta \frac{\cosh r}{r(F^2\cosh^2 r - 1)}(-\sin rX_2)$$

$$+F^2 \tan\beta\left\{\sum_{n=2}^{\infty} \frac{\cos y_n}{y_n(F^2\cos^2 y_n - 1)}(\mathrm{e}^{-|X_2|y_n} - \mathrm{e}^{-|X_3|y_n})\right\},$$

$$l\cot\alpha < x < l(\cot\alpha + \cot\beta); \tag{6.6c}$$

and

$$\eta(x)=\frac{F^2\tan\alpha}{2(F^2-1)}(|X_1|-|X_2|)+F^2\tan\alpha\left[\frac{\cosh r}{r(F^2\cosh^2 r-1)}(\sin rX_1-\sin rX_2)\right]$$
$$+F^2\tan\alpha\left\{\sum_{n=2}^{\infty}\frac{\cos y_n}{y_n(F^2\cos^2 y_n-1)}(\mathrm{e}^{-|X_2|y_n}-\mathrm{e}^{-|X_1|y_n})\right\}$$
$$+\frac{F^2\tan\beta}{2(F^2-1)}(|X_3|-|X_2|)+F^2\tan\beta\left[\frac{\cosh r}{r(F^2\cosh^2 r-1)}(\sin rX_3-\sin rX_2)\right]$$
$$+F^2\tan\beta\left\{\sum_{n=2}^{\infty}\frac{\cos y_n}{y_n(F^2\cos^2 y_n-1)}(\mathrm{e}^{-|X_2|y_n}-\mathrm{e}^{-|X_3|y_n})\right\},$$
$$x>l(\cot\alpha+\cot\beta). \tag{6.6d}$$

7. Summary and discussion

The two-dimensional flow of an ideal fluid over a nonuniform bottom has been investigated. The linearized theory derived under the assumption that the height of the triangle l is small compared to the channel depth h, predicts the existence of two different types of solutions.

For subcritical flow ($F^2<1$) a wave-free region is predicted upstream. It appears that there is a local disturbance immediately above the summit of the triangle, followed by a regular wave train downstream, as shown in Fig. 2 to Fig. 5. The amplitude of the local disturbance depends greatly on F^2 and the height l, as it is clear in Fig. 6, but it is not affected by the shape of the triangle. The amplitude of the downstream waves depends on F^2, the height l, and the shape of the triangle.

For supercritical flow ($F^2>1$), an essentially wave-free region is obtained upstream, the free surface consists of a single depression above the summit of the triangle, as shown in Fig. 7 to Fig.

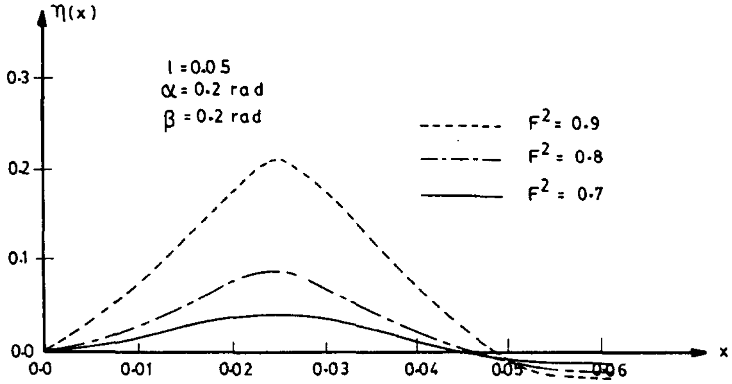

Fig. 2. Local disturbances versus x for different values of Froude number corresponding to $l=0.05$, $\alpha=0.2$, and $\beta=0.2$.

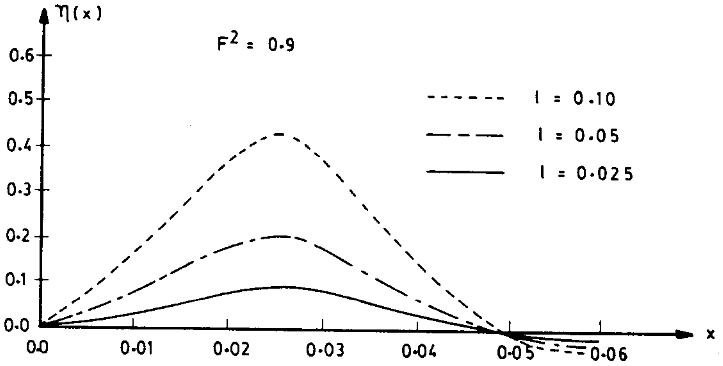

Fig. 3. Three shapes for the local disturbances for different values of the height l corresponding to $F^2 = 0.9$.

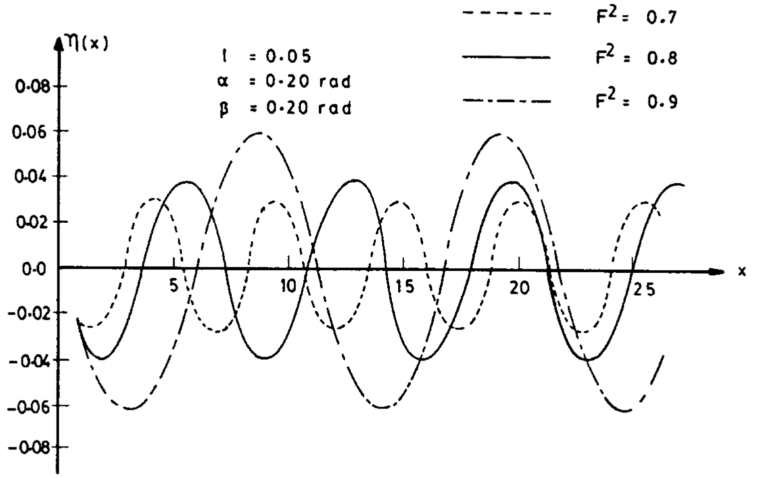

Fig. 4. Downstream waves for different values of Froude number corresponding to $l = 0.05$, $\alpha = 0.2$, and $\beta = 0.2$.

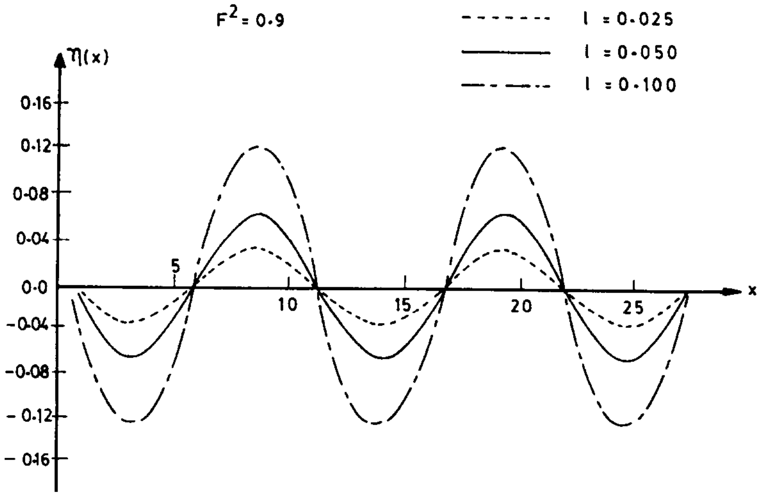

Fig. 5. Downstream waves for different values of the height l and $F^2 = 0.9$.

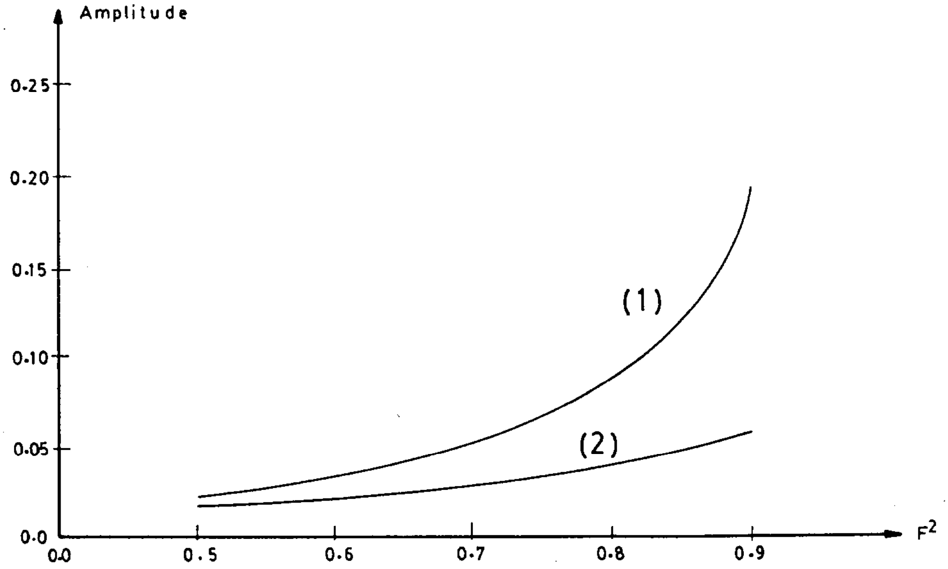

Fig. 6. Amplitude of waves against F^2, (1) for local disturbance, (2) for downstream.

9. The amplitude of the depression depends on F^2, the height l, and the shape of the triangle. From Fig. 10 it is observed that the depression (in magnitude) increases linearly with the height of the triangle l.

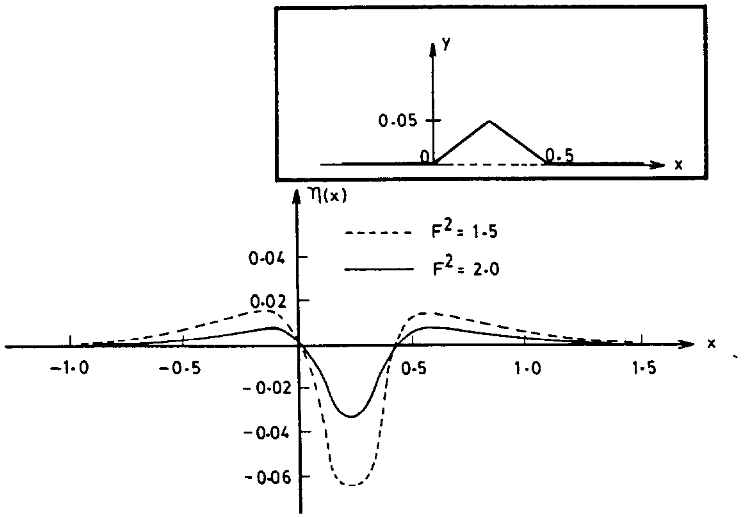

Fig. 7. Two free-surface profiles for different values of Froude number corresponding to $l = 0.05$, $\alpha = 0.2$, and $\beta = 0.2$.

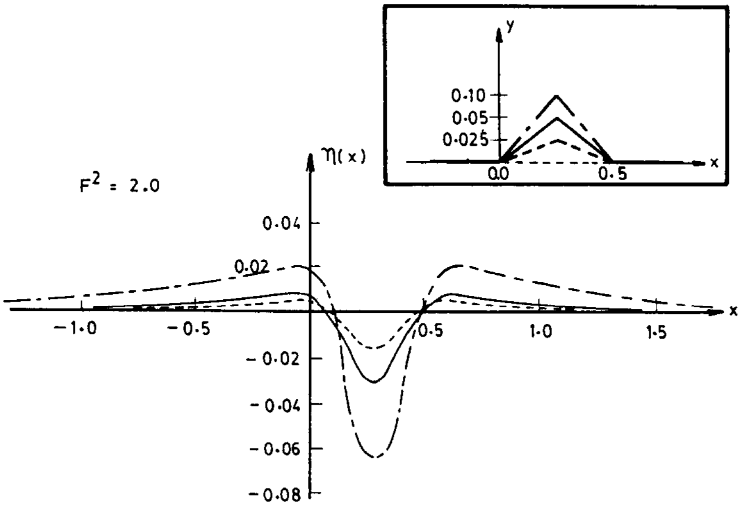

Fig. 8. Three free-surface profiles for different values of the height l corresponding to $F^2 = 2.0$.

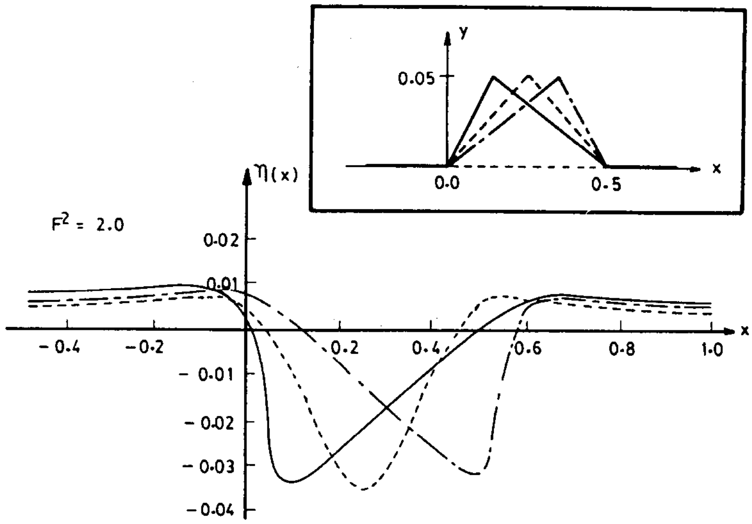

Fig. 9. Three free-surface profiles for three different shapes of the bottom for the same height $l = 0.05$ and corresponding $F^2 = 2.0$.

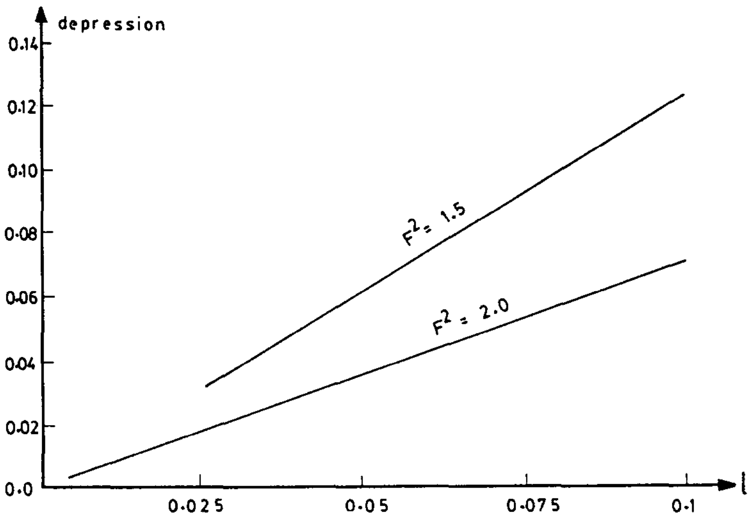

Fig. 10. Depression against the bottom height l for different values of F^2.

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